Reduction Formulas for ω_N:

- $\omega_b^a = \omega_d^c$ if $\frac{a}{b} = \frac{c}{d}$

Examples: $\omega_6^2 = \omega_3^1$ $\quad$ $\omega_8^4 = \omega_2^1$

$\omega_6^4 = \omega_3^2$ $\quad$ $\omega_{12}^9 = \omega_4^3$

("ω_b^a rotates $\frac{a}{b}$ around circle in $\mathbb{C}$")

- $\omega_N^{N+k} = \omega_N^k$ $\leftrightarrow$ So $\omega_N^{-k} = \omega_N^{N-k}$

Examples: $\omega_6^8 = \omega_6^2$ $\quad$ $\omega_6^{218} = \omega_6^2$

$\omega_6^{13} = \omega_6$ $\quad$ $\omega_6^{-2} = \omega_6^4$

("$\omega_N^N = 1$ one full rotation around circle")

Application to ω_{12}^k:

switch Real & Im (ω_{4N} rule)

$\omega_{12}^0 = 1$	$\omega_{12}^4 = -\bar{\omega}_{12}^2$	$\omega_{12}^8 = -\omega_{12}^2$
ω_{12}^1	$\omega_{12}^5 = -\bar{\omega}_{12}^1$	$\omega_{12}^9 = -\omega_{12}^3 = -\omega_4^1$
$\omega_{12}^2 = \omega_6^1$	$\omega_{12}^6 = \omega_2^1 = -1$	$\omega_{12}^{10} = \bar{\omega}_{12}^2$
$\omega_{12}^3 = \omega_4^1$	$\omega_{12}^7 = -\omega_{12}^1$	$\omega_{12}^{11} = \bar{\omega}_{12}$

- $\omega_N^{-k} = \bar{\omega}_N^k$ $\leftrightarrow$ So $\omega_N^{N-k} = \omega_N^{-k} = \bar{\omega}_N^k$

Examples: $\omega_6^{-2} = \bar{\omega}_6^2$

$\omega_6^4 = \omega_6^{-2} = \bar{\omega}_6^2$

$\omega_9^6 = \omega_9^{-3} = \bar{\omega}_9^3$

("$\bar{\omega}_N$ rotates in reverse direction")

- $\omega_{2N}^{N+k} = -\omega_{2N}^k$

Examples: $\omega_6^4 = -\omega_6^1$ $\quad$ $\omega_8^7 = -\omega_8^3$

$\omega_{12}^8 = -\omega_{12}^2$ $\quad$ $\omega_{10}^9 = -\omega_{10}^4$

("$\omega_{2N}^N = \omega_2^1 = -1$ rotate to opposite side")

- $\omega_{2N}^{N-k} = -\bar{\omega}_{2N}^k$ (because $\omega_{2N}^{N-k} = -\omega_{2N}^{-k} = -\bar{\omega}_{2N}^k$)

Examples: $\omega_6^2 = -\bar{\omega}_6$ $\quad$ ($\omega_6^2 = \omega_6^{3-1} = -\bar{\omega}_6^1$)

$\omega_{12}^4 = -\bar{\omega}_{12}^2$ $\quad$ ($\omega_{12}^4 = \omega_{12}^{6-2} = -\bar{\omega}_{12}^2$)

$\omega_{10}^3 = -\bar{\omega}_{10}^2$ $\quad$ ($\omega_{10}^3 = \omega_{10}^{5-2} = -\bar{\omega}_{10}^2$)

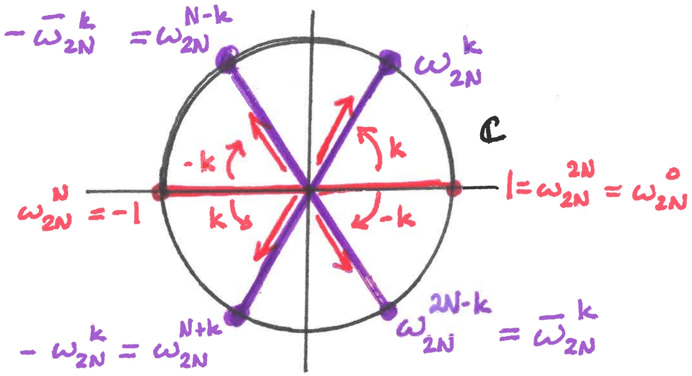
Note on changes

$\omega_N^{N+k} = \omega_N^{k}$ ← same Real and Im part

$\omega_N^{N-k} = \bar{\omega}_N^{k}$ ← switch sign on Im part

$\omega_{2N}^{N+k} = -\omega_{2N}^{k}$ ← switch sign on both parts

$\omega_{2N}^{N-k} = -\bar{\omega}_{2N}^{k}$ ← switch sign on Real part



Example:

ω_6^1

$\omega_6^2 = \omega_6^{3-1} = -\bar{\omega}_6^1$ ← switch sign on Real part

$\omega_6^4 = \omega_6^{3+1} = -\omega_6^1$ ← switch sign on both parts

$\omega_6^5 = \omega_6^{6-1} = \bar{\omega}_6^1$ ← switch sign on Im part.

Example:

ω_{12}^1

$\omega_{12}^5 = \omega_{12}^{6-1} = -\bar{\omega}_{12}^1$ ← switch sign on Real part.

$\omega_{12}^7 = \omega_{12}^{6+1} = -\omega_{12}^1$ ← switch sign on both parts

$\omega_{12}^{11} = \omega_{12}^{12-1} = \bar{\omega}_{12}^1$ ← switch sign on Im part.

Connection to Fast Fourier Transform

Basically this same pattern appears in Fast Fourier

- Split $\underline{f}$ into $\underline{f}_{even}$ & $\underline{f}_{odd}$
- Compute $\mathcal{F}_k\{\underline{f}_{even}\}$ & $\bar{\omega}_{2N}^{k}\,\mathcal{F}_k\{\underline{f}_{odd}\}$
- Combine

$$\mathcal{F}_k\{\underline{f}\} = \frac{1}{2}\left[\mathcal{F}_k\{\underline{f}_{even}\} + \bar{\omega}_{2N}^{k}\,\mathcal{F}_k\{\underline{f}_{odd}\}\right]$$

$$\mathcal{F}_{N-k}\{\underline{f}\} = \overline{\mathcal{F}_{N+k}\{\underline{f}\}}$$

$$\mathcal{F}_{N+k}\{\underline{f}\} = \frac{1}{2}\left[\mathcal{F}_k\{\underline{f}_{even}\} - \bar{\omega}_{2N}^{k}\,\mathcal{F}_k\{\underline{f}_{odd}\}\right]$$

$$\mathcal{F}_{2N-k}\{\underline{f}\} = \overline{\mathcal{F}_k\{\underline{f}\}}$$